Singular Coverings and Non-Uniform Notions of Closed Set Computability

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The empty set of course contains no computable point. On the other hand, surprising results due to Zaslavskiĭ, Tseĭtin, Kreisel, and Lacombe have asserted the existence of *non*-empty co-r.e. closed sets devoid of computable points: sets which are even 'large' in the sense of positive Lebesgue measure.

This leads us to investigate for various classes of computable real subsets whether they necessarily contain a (not necessarily effectively findable) computable point.

1 Introduction

A discrete set A, for example a subset of $\{0,1\}^*$ or $\mathbb{N}$, is naturally called r.e. (i.e. semi-decidable) if a Turing machine can enumerate the members of (equivalently: terminate exactly for inputs from) A. The corresponding notions for open subsets of reals [Laco57, Laco58, Weih00] amount to the following

Definition 1.1 *Fix a dimension $d \in \mathbb{N}$. An open subset $U \subseteq \mathbb{R}^d$ is called* r.e. *if and only if a Turing machine can enumerate rational centers $\vec{q}_n \in \mathbb{Q}^d$ and radii $r_n \in \mathbb{Q}$ of open Euclidean balls $B^\circ(\vec{q}, r) = \big\{\vec{x} \in \mathbb{R}^d : \|\vec{x} - \vec{q}\| < r\big\}$ exhausting U.*

*A real vector $\vec{x} \in \mathbb{R}^d$ is (Cauchy– or ρ^d–)*computable *if and only if a Turing machine can generate a sequence $\vec{q}_n \in \mathbb{Q}^d$ of rational approximations converging to $\vec{x}$* fast *in the sense that $\|\vec{x} - \vec{q}_n\| \leq 2^{-n}$.*

Notice that an open real subset is r.e. if and only if membership "$\vec{x} \in U$" is semi-decidable with respect to $\vec{x}$ given by fast convergent rational approximations; see for instance [Zieg04, LEMMA 4.1c].

1.1 Singular Coverings

A surprising result due to E. SPECKER implies that the (countable) set $\mathbb{R}_c$ of computable reals is contained in an r.e. open *proper* subset U of $\mathbb{R}$: In his work [Spec59] he constructs a computable function $f : [0,1] \to [0, \frac{1}{36}]$ attaining its maximum $\frac{1}{36}$ in no computable point; hence $U := (-\infty, 0) \cup f^{-1}[(-1, \frac{1}{36})] \cup (1, \infty)$ has the claimed properties, see for example [Weih00, THEOREM 6.2.4.1]. This was strengthened in [ZaTs62, KrLa57] to the following

Fact 1.2 *For any $\epsilon > 0$ there exists an r.e. open set $U_\epsilon \subseteq \mathbb{R}$ of Lebesgue measure $\lambda(U_\epsilon) < \epsilon$ containing all computable real numbers.*

Proof. See [Kush84, SECTION 8.1] or [Bees85, SECTION IV.6] or [Weih00, THEOREM 4.2.8]. □

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The significance of this improvement thus lies in the constructed U_ϵ intuitively being very 'small': it misses many non-computable points. On the other hand it is folklore that a certain smallness is also necessary: Every r.e. open $U \subsetneq \mathbb{R}$ covering $\mathbb{R}_c$ *must* miss uncountably many non-computable points. Put differently, an at most countable non-empty closed real subset must, if its complement is r.e., contain a computable point; see Observation 2.4 below.

This leads the present work to study further natural effective classes of closed Euclidean sets with respect to the question whether they contain a computable point. But let us start with reminding of the notion of

2 Computability of Closed Subsets

Decidability of a discrete set $A \subseteq \mathbb{N}$ amounts to computability of its characteristic function

$$\mathbf{1}_A(x) \;=\; 1 \quad \text{if} \quad x \in A, \qquad \mathbf{1}_A(x) \;=\; 0 \quad \text{if} \quad x \notin A \enspace .$$

Literal translation to the real number setting fails of course due to the continuity requirement; instead, the characteristic function is replaced by the continuous distance function

$$\operatorname{dist}_A(x) \;=\; \inf\big\{\|x-a\| : a \in A\big\}$$

which gives rise to the following natural notions [BrWe99], [Weih00, COROLLARY 5.1.8]:

Definition 2.1 *Fix a dimension $d \in \mathbb{N}$. A closed subset $A \subseteq \mathbb{R}^d$ is called*

- r.e. *if and only if $\operatorname{dist}_A : \mathbb{R}^d \to \mathbb{R}$ is upper computable;*
- co-r.e. *if and only if $\operatorname{dist}_A : \mathbb{R}^d \to \mathbb{R}$ is lower computable;*
- recursive *if and only if $\operatorname{dist}_A : \mathbb{R}^d \to \mathbb{R}$ is computable.*

Lower computing $f : \mathbb{R}^d \to \mathbb{R}$ amounts to the output, given a sequence $(\vec{q}_n) \in \mathbb{Q}^d$ with $\|\vec{x} - \vec{q}_n\| \leq 2^{-n}$, of a sequence $(p_m) \in \mathbb{Q}$ with $f(\vec{x}) = \sup_m p_m$. This intuitively means approximating f from below and is also known as $(\rho^d, \rho_<)$–computability with respect to standard real representations ρ and $\rho_<$; confer [Weih00, SECTION 4.1] or [WeZh00]. A closed set is co-r.e. if and only if its complement (an open set) is r.e. in the sense of Definition 1.1 [Weih00, SECTION 5.1]. Several other reasonable notions of closed set computability have turned out as equivalent to one of the above; see [BrWe99] or [Weih00, SECTION 5.1]: recursivity for instance is equivalent to *Turing location* [GeNe94] as well as to being simultaneously r.e. and co-r.e. This all has long confirmed Definition 2.1 as natural indeed.

2.1 Non-Empty Co-R.E. Closed Sets without Computable Points

Like in the discrete case, r.e. and co-r.e. are logically independent also for closed real sets:

Example 2.2 For $x := \sum_{n \in H} 2^{-n}$ (where $H \subseteq \mathbb{N}$ denotes the Halting Problem), the compact interval $I_< := [0,x] \subseteq \mathbb{R}$ is r.e. but not co-r.e.; and $I_> := [x,1]$ is co-r.e. but not r.e. □

Notice that both intervals have continuum cardinality and include lots of computable points. As a matter of fact, it is a well-known

Fact 2.3 *Let $A \subseteq \mathbb{R}^d$ be r.e. closed and non-empty. Then A contains a computable point* [Weih00, EXERCISE 5.1.13b].

More precisely, closed $\emptyset \neq A \subseteq \mathbb{R}^d$ is r.e. if and only if $A = \overline{\{\vec{x}_1, \ldots, \vec{x}_n, \ldots\}}$ for some computable sequence $(\vec{x}_n)_n$ of real vectors [Weih00, LEMMA 5.1.10].

A witness of (one direction of) logical independence stronger than $I_>$ is thus a non-empty co-r.e. closed set A devoid of computable points: $A \subseteq [0,1] \setminus \mathbb{R}_c$. For example every singular covering U_ϵ with $\epsilon < 1$ from Section 1.1 due to [ZaTs62, KrLa57] gives rise to an instance $A_\epsilon := [0,1] \setminus U_\epsilon$ even of positive Lebesgue measure $\lambda(A) > 1 - \epsilon$, and thus of continuum cardinality. Conversely, it holds

Observation 2.4 *Every non-empty co-r.e. closed set of cardinality strictly* less *than that of the continuum* does *contain computable points.*

Notice that this claim also covers putative cardinalities between $\aleph_0$ and $2^{\aleph_0} = \mathfrak{c}$ i.e. does not rely on the Continuum Hypothesis.

In a finite set, every point is isolated; in this case the claim thus follows from the well-known

Fact 2.5 a) *Let $A \subseteq \mathbb{R}^d$ be co-r.e. closed and suppose there exist $\vec{a}, \vec{b} \in \mathbb{Q}^d$ such that $A \cap [\vec{a}, \vec{b}] = \{\vec{x}\}$ (where $[\vec{a}, \vec{b}] := \prod_{i=1}^d [a_i, b_i]$). Then, $\vec{x}$ is computable.*

b) *A perfect subset $A \subseteq X$ (of $X = \mathbb{R}^d$ or of $X = \{0,1\}^\omega$), i.e. one which coincides with the collection A' of its limit points,*

$$A' \quad := \quad \left\{ \vec{x} \in X \;\middle|\; \forall n \exists \vec{a} \in A : \; 0 < |\vec{a} - \vec{x}| < 1/n \right\} \ ,$$

is either empty or of continuum cardinality.

See for instance [BrWe99, PROPOSITION 3.6] and [Kech95, COROLLARY 6.3].

Proof (Observation 2.4). Suppose that A has cardinality strictly less than that of the continuum. Then $A \neq A'$ by Fact 2.5b). On the other hand, A contains A' because it is closed. Hence the difference $A \setminus A' \neq \emptyset$ holds and consists of isolated points which are computable by Fact 2.5a). □

So every non-empty co-r.e. closed real set $A \subseteq [0,1]$ devoid of computable points must necessarily be of continuum cardinality. On the other hand, Fact 1.2 yields such sets with positive Lebesgue measure $\lambda(A) > 0$. In view of (and in-between) the strict[1] chain of implications

$$\text{nonempty interior} \quad \substack{\Rightarrow \\ \not\Leftarrow} \quad \text{positive measure} \quad \substack{\Rightarrow \\ \not\Leftarrow} \quad \text{continuum cardinality}$$

we make the following[2]

Remark 2.6 *There exists a non-empty co-r.e. closed real subset of measure zero without computable points.*

This is different from [Kush84, SECTION 8.1] which considers

- coverings of $(0,1)$ having measure *strictly* less than 1
- by disjoint enumerable 'segments', that is *closed* intervals $[a_n, b_n]$,
- or by enumerable open intervals (a_n, b_n) as in Definition 1.1, however in terms of the *accumulated* length $\sum_n (b_n - a_n)$, that is counting interval overlaps doubly [Kush84, THEOREM 8.5].

Proof (Remark 2.6). Take a subset A of Cantor space with these properties and consider its image $\tilde{A}$ under the canonical embedding

$$\{0,1\}^\omega \;\ni\; (b_n) \;\mapsto\; \sum_n b_n 2^{-n} \;\in\; [0,1] \ .$$

Notice that this mapping, restricted to A, is indeed injective because only dyadic rationals have a non-unique binary expansion; and in fact two of them, both of which are decidable. Therefore

- $\tilde{A}$ has continuum cardinality but, being contained in Cantor's Middle Third set, has measure zero.
- The enumeration of open balls in $\{0,1\}^\omega$ exhausting A's complement translates to one exhausting $[0,1] \setminus \tilde{A}$.
- Suppose $x \in \tilde{A}$ were computable. Then x has decidable binary expansion [Weih00, THEOREM 4.1.13.2], contradicting that all elements of $\tilde{A}$ arise from uncomputable binary sequences $(b_n) \in A$.

□

[1] Consider for instance the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ and CANTOR's uncountable Middle Third set, respectively.

[2] We are grateful to a careful anonymous referee for indicating this simple solution to a question raised in an earlier version of this work.

2.2 Computability on Classes of Closed Sets of Fixed Cardinality

Observation 2.4 and Fact 2.5a) are non-uniform claims: they assert a computable point in A to *exist* but not that it can be 'found' effectively. Nevertheless, a uniform version of Fact 2.5a) does hold under the additional hypothesis that $\vec{a}$ and $\vec{b}$ are known; compare [Weih00, EXERCISE 5.2.3] reported as Lemma 2.8a) below. The present section investigates whether and to what extend this result can be generalized towards Observation 2.4 and, to this end, considers the following representations for (classes of) closed real sets of fixed cardinality:

Definition 2.7 *For $d \in \mathbb{N}$ and closed $A \subseteq \mathbb{R}^d$,*

- $\psi^d_<$ *encodes A as a $[\rho^d \to \rho_>]$–name of* dist_A*;*
- $\psi^d_>$ *encodes A as a $[\rho^d \to \rho_<]$–name of* dist_A

in the sense of [WeZh00].

Write $\mathcal{A}^d_N := \{A \subseteq [0,1]^d \text{ closed} : \mathrm{Card}(A) = N\}$ for the hyperspace of compact sets having cardinality exactly N, where $N \leq \mathfrak{c}$ denotes a cardinal number. Equip $\mathcal{A}^d_N$ with restrictions $\psi^d_<|^{\mathcal{A}^d_N}$ and $\psi^d_>|^{\mathcal{A}^d_N}$ of the above representations.

If $N \leq \aleph_0$, we furthermore can encode $A \subseteq [0,1]^d$ of cardinality N (closed or not) by the join of the ρ^d–names of the N elements constituting A, listed in arbitrary order[3]*. This representation shall be denoted as $(\rho^d)^{\sim N}$.*

Let us first handle finite cardinalities:

Lemma 2.8 *Fix $d \in \mathbb{N}$.*

a) $\psi^d_<\big|^{\mathcal{A}^d_1} \;\equiv\; (\rho^d)^{\sim 1} \;\equiv\; \psi^d_>\big|^{\mathcal{A}^d_1}$

b) *For $2 \leq N \in \mathbb{N}$, it holds $\psi^d_<\big|^{\mathcal{A}^d_N} \;\equiv\; (\rho^d)^{\sim N} \;\precneqq\; \psi^d_>\big|^{\mathcal{A}^d_N}$*

c) *For $N \in \mathbb{N}$, $A \in \mathcal{A}^d_N$ is $\psi^d_<$–computable if and only if it is $\psi^d_>$–computable.*

In particular, [Weih00, EXAMPLE 5.1.12.1] generalizes to arbitrary finite sets:

Corollary 2.9 *A finite subset A of $\mathbb{R}^d$ is r.e. if and only if A is co-r.e. if and only if every point in A is computable.*

Proof (Lemma 2.8). a) Confer [Weih00, EXERCISE 5.2.3].

b) The reductions "$(\rho^d)^{\sim N} \preceq \psi^d_<$" and "$(\rho^d)^{\sim N} \preceq \psi^d_>$" follow from induction on N via Claim a) and [Weih00, THEOREM 5.1.13.1]. "$\psi_>|^{\mathcal{A}_2} \not\preceq (\rho)^{\sim 2}$" can be seen easily based on a straight-forward discontinuity argument. For "$\psi^d_<|^{\mathcal{A}^d_N} \preceq (\rho^d)^{\sim N}$", recall that a $\psi^d_<$–name for A is (equivalent to) the ρ^d–names of countably infinitely many points $\vec{x}_m \in A$ dense in A [Weih00, LEMMA 5.1.10]. Since $A = \{\vec{a}_1, \ldots, \vec{a}_N\}$ is finite, there exist $m_1, \ldots, m_N \in \mathbb{N}$ such that $\vec{x}_{m_n} = \vec{a}_n$ for $n = 1, \ldots, N$; equivalently: $\vec{x}_{m_i} \neq \vec{x}_{m_j}$ for $i \neq j$. The latter condition also yields a way to effectively find such indices $m_1, \ldots, m_N$, regarding that *in*equality is semi-decidable. Once found, the ρ^d–names of $\vec{x}_{m_1}, \ldots, \vec{x}_{m_N}$ constitute a $(\rho^d)^{\sim N}$–name of A.

c) That $\psi^d_<$–computability implies $\psi^d_>$–computability follows from a). For the converse, the case $N = 1$ is covered in Claim a). In case $N > 1$ exploit that points of A lie isolated; that is, there exist closed rational cubes $Q_n := [\vec{a}_n, \vec{b}_n]$, $n = 1, \ldots, N$ containing exactly one element of A each. By storing their finitely many coordinates, a Type-2 machine is able to $\psi^d_>$–compute the N closed one-element sets $A \cap Q_n$. We have thus effectively reduced to the case $N = 1$.

□

The case of countably infinite closed sets:

Lemma 2.10 a) *In the definition of $(\rho^d)^{\sim \aleph_0}$, it does not matter whether each element $\vec{x}$ of A is required to occur exactly once or at least once.*

[3] see also Lemma 2.10a)

b) *It holds* $(\rho^d)^{\sim\aleph_0}\big|^{\mathcal{A}^d_{\aleph_0}} \;\not\preceq\; \psi^d_<\big|^{\mathcal{A}^d_{\aleph_0}}$.

c) *There exists a countably infinite r.e. closed set* $A \subseteq [0,1]$ *which is neither* $\rho^{\sim\aleph_0}$*–computable nor co-r.e.*

d) *There is a countably infinite co-r.e. but not r.e. closed set* $B \subseteq [0,1]$.

Proof. a) [Weih00, EXERCISE 4.2.3] holds uniformly and extends from $\mathbb{R}$ to $\mathbb{R}^d$: Let vectors $\vec{x}_m \in \mathbb{R}^d$ be given by ρ^d–names, $m \in \mathbb{N}$. Based on semi-decidable *in*equality "$\vec{x}_m \neq \vec{x}_n$", we can employ dovetailing to identify infinitely many distinct ones like in the proof of Lemma 2.8a). However infinitely many may be not enough: some care is required to find *all* of them. To this end, take the given $\vec{q}_{m,n} \in \mathbb{Q}^d$ with $\|\vec{q}_{m,n} - \vec{x}_m\|_2 < 2^{-n}$ and suppose $M_n \subseteq \mathbb{N}$ is a finite set of indices of vectors already identified as distinct, that is, with $\vec{x}_m \neq \vec{x}_{m'}$ for $m, m' \in M_n$, $m \neq m'$. Then, in phase $n+1$, consider the smallest (!) index m_{n+1} newly recognized as different from all $\vec{x}_m$, $m \in M_n$:

$$M'_n \;:=\; \big\{ m' \in \mathbb{N} \setminus M_n \;\big|\; \forall m \in M_n : \; B^\circ\big(\vec{q}_{m,n}, 2^{-n}\big) \;\cap\; B^\circ\big(\vec{q}_{m',n}, 2^{-n}\big) \;= \emptyset \big\},$$

$$m_{n+1} \;:=\; \min M'_n, \qquad M_{n+1} := M_n \cup \{m_{n+1}\}$$

if $M'_n \neq \emptyset$, otherwise $M_{n+1} := M_n$. Notice that an element of M'_n re-appears in M'_{n+1} unless it was the minimal one: if balls are disjoint, they remain so when reducing the radius. Let's argue further to assert correctness of this algorithm: By prerequisite there are infinitely many distinct vectors among the $(\vec{x}_m)$, hence $M'_n \neq \emptyset$ infinitely often, yielding a sequence $(\vec{x}_{m_n})_n$ (not a *sub*sequence since that would require (m_n) to be increasing) of distinct elements; in fact of *all* of them: If $m' \in \mathbb{N}$ is such that $\vec{x}_{m'} \neq \vec{x}_m$ for all $m < m'$, then there exists some $n \in \mathbb{N}$ for which $m' \in M'_n$. By virtue of the above observation, m' eventually becomes the minimal element of some later $M'_{n'}$ and thus does occur in the output as index $m_{n'}$.

b) The positive part of the claim follows from [Weih00, LEMMA 5.1.10], asserting that a sequence of real vectors dense in closed A yields a $\psi_<$–name of A; whereas (negative) unreducibility is a consequence of Claim c).

c) Let $(x_n)_n$ denote a Specker Sequence, that is, a computable sequence converging (non-effectively) from below to the uncomputable real $x_\infty = \sum_{n\in H} 2^{-n}$. Then $(x_n)_n$ is dense in closed $A := \{\vec{x}_n : n \in \mathbb{N}\} \cup \{x_\infty\} \subseteq [0,1]$, hence the latter $\psi_<$–computable. But $x_\infty = \max A$ is $\rho_>$–uncomputable, therefore A cannot be $(\rho)^{\sim\aleph_0}$–computable; nor $\psi_>$–computable [Weih00, LEMMA 5.2.6.2].

d) For the Halting Problem $H \subseteq \mathbb{N}$ consider the closed set $B := \{0\} \cup \{2^{-n} : n \not\in H\} \subseteq [0,1]$. The open rational intervals $(2^{-n-1}, 2^{-n})$ for all $n \in \mathbb{N}$ and, for $n \in H$ by semi-decidability, $(2^{-n-1}, 3 \cdot 2^{-n-1})$ together exhaust exactly $(0,1) \setminus A$; this enumeration thus establishes $\psi_>$–computability of B. $\psi_<$–computability fails due to [Weih00, EXERCISE 5.1.5].

□

3 Closed Sets and Naively Computable Points

A notion of real computability weaker than that of Definition 1.1 is given in the following

Definition 3.1 *A real vector* $\vec{x} \in \mathbb{R}^d$ *is* naively computable *(also called* recursively approximable*) if a Turing machine can generate a sequence* $\vec{q}_n \in \mathbb{Q}^d$ *with* $\vec{x} = \lim_n \vec{q}_n$ *(i.e. converging but not necessarily fast).*

A real point is naively computable if and only if it is Cauchy–computable *relative* to the Halting oracle $H = \emptyset'$, see [Ho99, THEOREM 9] or [ZhWe01].

Section 2.1 asked whether *certain* non-empty co-r.e. closed sets contain a Cauchy–computable element. Regarding naively computable elements, it holds

Proposition[4] 3.2 Every *non-empty co-r.e. closed set* $A \subseteq \mathbb{R}^d$ *contains a naively computable point* $\vec{x} \in A$.

[4] A simple reduction to the counterpart of this claim for Baire space [CeRe98, THEOREM 2.6(c)] does not seem feasible because, according to [Weih00, THEOREM 4.1.15.1], there exists no *total* (compact or not) representation equivalent to ρ.

W.l.o.g. A may be presumed compact by proceeding to $A \cap [\vec{u}, \vec{v}]$ for appropriate $\vec{u}, \vec{v} \in \mathbb{Q}^d$ [Weih00, THEOREM 5.1.13.2]. In 1D one can then explicitly choose $x = \max A$ according to [Weih00, LEMMA 5.2.6.2]. For higher dimensions we take a more implicit approach and apply Lemma 3.4a) to the following relativization of Fact 2.3:

Scholium[5] **3.3** *Let non-empty $A \subseteq \mathbb{R}^d$ be r.e. closed* relative *to $\mathcal{O}$ for some oracle $\mathcal{O}$. Then A contains a point computable* relative *to $\mathcal{O}$.*

Lemma 3.4 *Fix closed $A \subseteq \mathbb{R}^d$.*

a) *If A is co-r.e., then it is also r.e. relative to $\emptyset'$.*

b) *If A is r.e., then it is also co-r.e. relative to $\emptyset'$.*

These claims may follow from [Brat05, Gher06]. However for purposes of self-containment we choose to give a direct

Proof. Recall [Weih00, DEFINITION 5.1.1] that a $\psi^d_>$–name of A is an enumeration of all closed rational balls $\overline{B}$ disjoint from A; whereas a $\psi^d_<$–name enumerates all open rational balls B° intersecting A. Observe that

$$\begin{aligned} B^\circ \cap A \neq \emptyset \quad &\Leftrightarrow \quad \exists n \in \mathbb{N} : \overline{B}_{-1/n} \cap A \neq \emptyset \\ \overline{B} \cap A = \emptyset \quad &\Leftrightarrow \quad \exists n \in \mathbb{N} : B^\circ_{+1/n} \cap A = \emptyset \end{aligned} \tag{1}$$

where $B_{\pm\epsilon}$ means enlarging/shrinking B by ϵ such that $B^\circ = \bigcup_n \overline{B}_{+1/n}$ and $\overline{B} = \bigcap_n B^\circ_{-1/n}$. Formally in 1D e.g. $(u, v)_{-\epsilon} := (u + \epsilon, v - \epsilon)$ in case $v - u > 2\epsilon$, $(u, v)_{-\epsilon} := \{\}$ otherwise. Under the respective hypothesis of a) and b), the corresponding right hand side of Equation (1) is obviously decidable relative to $\emptyset'$. □

A simpler argument might try to exploit [Ho99, THEOREM 9] that every $\rho_<$–computable single real y is, relative to $\emptyset'$, $\rho_>$–computable; and conclude by uniformity that (Definition 2.1) every $(\rho \to \rho_<)$–computable function $f : x \mapsto f(x) = y$ is, relative to $\emptyset'$, $(\rho \to \rho_>)$–computable. This conclusion however is wrong in general because even a relatively $(\rho \to \rho_>)$–computable f must be upper semi-continuous whereas a $(\rho \to \rho_<)$–computable one may be merely lower semi-continuous.

3.1 (In-)Effective Compactness

By virtue of the Heine–Borel and Bolzano–Weierstrass Theorems, the following properties of a real subset A are equivalent:

i) A is closed and bounded;

ii) every open rational cover $\bigcup_{n\in\mathbb{N}} B^\circ(\vec{q}_n, r_n)$ of A contains a finite sub-cover;

iii) any sequence $(\vec{x}_n)$ in A admits a subsequence $(\vec{x}_{n_k})$ converging within A.

Equivalence "i)⇔ii)" (Heine–Borel) carries over to the effective setting [Weih00, LEMMA 5.2.5] [BrWe99, THEOREM 4.6]. Regarding sequential compactness iii), a Specker Sequence (compare the proof of Lemma 2.10c) yields the counter-example of a recursive rational sequence in $A := [0, 1]$ having no recursive *fast* converging subsequence, that is, no computable accumulation point. This leaves the question whether every bounded recursive sequence admits an at least *naively* computable accumulation point. Simply taking the *largest* one (compare the proof of Proposition 3.2 in case $d = 1$) does not work in view of [ZhWe01, THEOREM 6.1]. Also effectivizing the Bolzano–Weierstraß selection argument yields only an accumulation point computable relative to $\emptyset''$:

Observation 3.5 *Let $(x_n) \subseteq [0, 1]$ be a bounded sequence. For each $m \in \mathbb{N}$ choose $k = k(m) \in \mathbb{N}$ such that there are infinitely many n with $x_n \in B^\circ(x_k, 2^{-m})$. Boundedness and pigeonhole principle, inductively for $m = 1, 2, \ldots$, assert the existence of smaller and smaller (length 2^{-m}) sub-intervals each containing infinitely many members of that sequence:*

$$\exists a, b \in \mathbb{Q} \;\; \forall N \;\; \exists n \geq N : \quad x_n \in (a, b) \;\wedge\; |b - a| \leq 2^{-m} \quad . \tag{2}$$

This is a Σ_3–formula; and thus semi-decidable relative *to $\emptyset''$, see for instance* [Soar87, POST'S THEOREM §IV.2.2].

[5] A scholium is "*a note amplifying a proof or course of reasoning, as in mathematics*" [Morr69]

In fact $\emptyset''$ is the best possible as we establish, based on Section 3.2,

Theorem 3.6 *There exists a recursive rational sequence* $(x_n) \subseteq [0,1]$ *containing no* naively *computable accumulation point.*

This answers a recent question in Usenet [Lagn06]. The sequence constructed is rather complicated—and must be so in view of the following counter-part to Fact 2.5a) and Observation 2.4:

Lemma 3.7 *Let* $(x_n) \subseteq [0,1]^d$ *be a computable real sequence and let* A *denote the set of its accumulation points.*

a) *Every isolated point* x *of* A *is naively computable.*

b) *If* $\operatorname{Card}(A) < \mathfrak{c}$*, then* A *contains a naively computable point.*

Proof. A is closed non-empty and thus, if in addition free of isolated points, perfect; so b) follows from a). Let $\{x\} = A \cap [u,v] = A \cap (r,s)$ with rational $u < r < s < v$. A subsequence (x_{n_m}) contained in (r,s) will then necessarily converge to x. Naive computability of x thus follows from selecting such a subsequence effectively: Iteratively for $m = 1, 2, \ldots$ use dove-tailing to search for (and, as we know it exists, also find) some integer $n_m > n_{m-1}$ with "$x_{n_m} \in (r,s)$". The latter property is indeed semi-decidable, for instance by virtue of [Zieg04, LEMMA 4.1c]. □

We have just been pointed out [Zhen07] that Theorem 3.6 can easily be proven by a standard diagonalization on an enumeration of all recursive rational sequences. However we prefer an alternative approach because the uniform Proposition 3.9 below may be of interest of its own. Indeed, Theorem 3.6 follows from applying to Proposition 3.9 a relativization of Fact 1.2 which is an easy consequence of for example the proof of [Weih00, THEOREM 4.2.8], namely

Scholium 3.8 *For any oracle* $\mathcal{O}$*, there exists a non-empty closed set* $A \subseteq [0,1]$ *co-r.e.* relative *to* $\mathcal{O}$*, containing no point Cauchy–computable* relative *to* $\mathcal{O}$*.*

3.2 Co-R.E. Closed Sets Relative to $\emptyset'$

[Ho99, THEOREM 9] has given a nice characterization of real numbers Cauchy–computable *relative* to the Halting oracle. We do similarly for co-r.e. closed real sets:

Proposition 3.9 *A closed subset* $A \subseteq \mathbb{R}^d$ *is* $\psi^d_>$*–computable* relative *to* $\emptyset'$ *if and only if it is the set of accumulation points of a recursive rational sequence or, equivalently, of an enumerable infinite subset of rationals.*

This follows (uniformly and for simplicity in case $d = 1$) from Claims a-e) of

Lemma 3.10 a) *Let closed* $A \subseteq \mathbb{R}$ *be co-r.e. relative to* $\emptyset'$*. Then there is a recursive double sequence of open rational intervals* $B^\circ_{m,n} = (u_{m,n}, v_{m,n})$ *and a (not necessarily recursive) function* $M : \mathbb{N} \to \mathbb{N}$ *such that*

i) $\forall N \in \mathbb{N}\ \forall m \geq M(N)\ \forall n \leq N :\quad B^\circ_{m,n} = B^\circ_{M(N),n} = \ldots =: B^\circ_{\infty,n}$ *(*$B^\circ_{m,1}, \ldots, B^\circ_{m,N}$ *each stabilizes beyond* $m \geq M(N)$*)*

ii) $A = \mathbb{R} \setminus \bigcup_n B^\circ_{\infty,n}$.

b) *From a double sequence* $B^\circ_{m,n}$ *of open rational intervals as in a i+ii), one can effectively obtain a rational sequence* (q_ℓ) *whose set of accumulation points coincides with* A*.*

c) *Given a rational sequence* (q_ℓ)*, a Turing machine can enumerate a subset* Q *of rational numbers having the same accumulation points. (Recall that a sequence may repeat elements but a set cannot.)*

d) *Given an enumeration of a subset* Q *of rational numbers, one can effectively generate a double sequence of open rational intervals* $B^\circ_{m,n}$ *satisfying i+ii) above where* A *denotes the set of accumulation points of* Q*.*

e) *If a double sequence of open rational intervals* $B^\circ_{m,n}$ *with i) is recursive, then the set* A *according to ii) is co-r.e. relative to* $\emptyset'$*.*

f) *Let* $N \in \mathbb{N}$, $\vec{u}_n, \vec{v}_n \in \mathbb{Q}^d$, *and* $\vec{x} \in \mathbb{R}^d$ *with* $\vec{x} \notin \bigcup_{n=1}^N (\vec{u}_n, \vec{v}_n)$. *Then, to every* $\epsilon > 0$, *there is some* $\vec{q} \in \mathbb{Q}^d \setminus \bigcup_{n=1}^N (\vec{u}_n, \vec{v}_n)$ *such that* $\|\vec{x} - \vec{q}\| \leq \epsilon$.

Proof. a) By [Weih00, LEMMA 5.1.10], $\psi_>$–computability of a closed set A implies (is even uniformly equivalent to) enumerability of open rational intervals B_n° with $A = \mathbb{R} \setminus \bigcup_n B_n^\circ$. By application of Limit Lemma (SHOENFIELD's? anyway, see for example [Soar87]) we conclude that $\psi_>$–computability *relative* to $\emptyset'$ implies (and follows from, see Item e) recursivity of a double sequence $B_{m,n}^\circ$ satisfying i) and ii).

b) Calculate $\tilde{N}(m) := \max\{N \leq m : B_{m,n}^\circ = B_{m+1,n}^\circ \ \forall n \leq N\}$ and let $(q_{\langle m,k \rangle})_k$ enumerate (without repetition) the set $\mathbb{Q} \setminus \bigcup_{n=1}^{\tilde{N}(m)} B_{m,n}^\circ$.

- For $x \notin A$, there exists N such that $x \in B_{\infty,N}^\circ$. By i), $\tilde{N}(m) \geq N$ for $m \geq M(N)$. Therefore $q_{\langle m,k \rangle} \in B_{\infty,N}^\circ$ can hold only for $m < M(N)$, i.e., finitely often; hence x is no accumulation point of (q_ℓ).
- Suppose $x \in A$, i.e. $x \notin \bigcup_n B_{\infty,n}^\circ \supseteq \bigcup_{n=1}^N B_{m,n}^\circ$ for every N and $m \geq M(N)$. In particular for $m \geq M(N)$, it holds $x \notin \bigcup_{n=1}^{\tilde{N}(m)} B_{m,n}^\circ$ and by construction plus Claim f) there is some k_m with $|q_{\langle m,k_m \rangle} - x| \leq 2^{-m}$. So x is an accumulation point of (q_ℓ).

c) Starting with $Q = \{\}$ add, inductively for each $\ell \in \mathbb{N}$, a rational number not yet in Q and closer to q_ℓ than 2^{-n}. Indeed finiteness of Q at each step asserts: $\exists p \in \mathbb{Q} \cap \big((q_\ell - 2^{-n}, q_\ell + 2^{-n}) \setminus Q\big)$.

d) Let $(B_{0,k}^\circ)_k$ denote an effective enumeration of all open rational intervals. Given $(q_\ell)_\ell$, calculate inductively for $m \in \mathbb{N}$ the subsequence $(B_{m+1,n}^\circ)_n$ of $(B_{m,n}^\circ)_n$ containing those intervals disjoint to $\{q_1, \ldots, q_m\}$.

- For x accumulation point of (q_ℓ) and $B_{0,k}^\circ$ an arbitrary open rational interval containing x, there is some $q_M \in B_{0,k}^\circ$. By construction, this $B_{0,k}^\circ$ will not occur in $(B_{m+1,n}^\circ)_n$ for $m \geq M$. Hence $x \in \mathbb{R} \setminus \bigcup_n B_{\infty,n}^\circ = A$.
- If x is contained in some interval $B_{0,k}^\circ$ which 'prevails' as $B_{\infty,n}^\circ$, it cannot contain any q_m by construction. Therefore x is no accumulation point.

e) Consider a Turing machine enumerating $(B_{m,n}^\circ)_{\langle m,n \rangle}$. $(B_{M(N),N}^\circ)_N$ is a $\psi_>$–name of A [Weih00, LEMMA 5.1.10]. Deciding for given $N, M \in \mathbb{N}$ whether "$B_{m,n}^\circ = B_{M,n}^\circ \ \ \forall m \geq M \ \ \forall n \leq N$" holds, is a Π_1–problem and thus possible relative to $\emptyset'$. With the help of this oracle, one can therefore compute $N \mapsto M(N)$ according to i).

f) If $\vec{x} \in \mathbb{Q}^d$ then let $\vec{q} := \vec{x}$. Otherwise $\vec{x}$ belongs to the open set $\mathbb{R} \setminus \bigcup_{n=1}^N [\vec{u}_n, \vec{v}_n]$ in which rational numbers lie dense.

□

4 Connected Components

Instead of asking whether a set contains a computable point, we now turn to the question whether it has a 'computable' connected component. Proofs here are more complicated but the general picture turns out rather similar to Section 2:

- If the co-r.e. closed set under consideration contains finitely many components, each one is again co-r.e. (Section 4.1).
- If there are countably many, some is co-r.e. (Section 4.2).
- There exists a compact co-r.e. set of which none of its (uncountably many) connected components is co-r.e. (Observation 4.3).

Recall that for a topological space X, the connected component $C(X,x)$ of $x \in X$ denotes the union over all connected subsets of X containing x. It is connected and closed in X. $C(X,x)$ and $C(X,y)$ either coincide or are disjoint.

Proposition 4.1 *Fix $d \in \mathbb{N}$.*

a) *Every (path[6]–) connected component of an r.e. open set is r.e. open.*
More precisely (and more uniformly) the following mapping is well-defined and $(\theta^d_<, \rho^d, \theta^d_<)$–computable:

$$\big\{(U,\vec{x}) : \vec{x} \in U \subseteq \mathbb{R}^d \textit{ open}\big\} \ \ni \ (U,\vec{x}) \ \mapsto \ C(U,\vec{x}) \ \subseteq \ \mathbb{R}^d \textit{ open.}$$

b) *The following mapping is well-defined and $(\psi^d_>, \rho^d, \psi^d_>)$–computable:*

$$\big\{(A,\vec{x}) : \vec{x} \in A \subseteq [0,1]^d \textit{ closed}\big\} \ \ni \ (A,\vec{x}) \ \mapsto \ C(A,\vec{x}) \ \subseteq \ [0,1]^d \textit{ closed.}$$

P r o o f. First observe that closedness of $C(A,\vec{x})$ in closed $A \subseteq [0,1]^d$ means compactness in $\mathbb{R}^d$. Similarly, open U is locally (even path-) connected, hence $C(U,\vec{x})$ open in U and thus also in $\mathbb{R}^d$.

a) Let $(B_1, B_2, \ldots, B_m, \ldots)$ denote a sequence of open rational balls exhausting U, namely given as a $\theta^d_<$–name of U. Since the non-disjoint union of two connected subsets is connected again,

$$\vec{x} \in B_{m_1} \ \ \wedge \ \ B_{m_i} \cap B_{m_{i+1}} \neq \emptyset \ \ \forall i < n \tag{3}$$

implies $B_{m_n} \subseteq C(U,\vec{x})$ for any choice of $n, m_1, \ldots, m_n \in \mathbb{N}$. Conversely, for instance by [Boto79, SATZ 4.14], there exists to every $\vec{y} \in C(U,\vec{x})$ a finite subsequence B_{m_i} $(i = 1,\ldots,n)$ satisfying (3) with $\vec{y} \in B_{m_n}$. Condition (3) being semi-decidable, one can enumerate all such subsequences and use them to exhaust $C(U,\vec{x})$. Nonuniformly, every connected component contains by openness a rational (and thus computable) 'handle' $\vec{x}$.

b) Recall the notion of a *quasi*-component [Kura68, §46.V]

$$Q(A,\vec{x}) \ := \bigcap_{S \in \mathcal{S}(A,\vec{x})} S, \qquad \mathcal{S} \ := \ \big\{S \subseteq A \ : \ S \text{ clopen in } A, \ \vec{x} \in S\big\} \tag{4}$$

where "clopen in A" means being both closed and open in the relative topology of A. That is, S is closed in $\mathbb{R}^d$, and so is $A \setminus S$! By the T_4 separation property (normal space), there exit disjoint open sets $U,V \subseteq \mathbb{R}^d$ such that $S \subseteq U$ and $A \setminus S \subseteq V$. In particular $S = A \cap \overline{U}$, $U \cap V = \emptyset$, and $A \subseteq U \cup V$:

$$\mathcal{S}(A,\vec{x}) \ = \ \big\{A \cap \overline{U} \ \big| \ U,V \subseteq \mathbb{R}^d \text{ open}, U \cap V = \emptyset, \ \vec{x} \in U, \ A \subseteq U \cup V\big\} \ . \tag{5}$$

Both U and V are unions from from the topological base of open rational balls; w.l.o.g. *finite* such unions by compactness of A: $U = \overline{B_1 \cup \ldots \cup B_n} = \overline{B}_1 \cup \ldots \cup \overline{B}_n$ and $V = B'_1 \cup \ldots \cup B'_m$. Therefore $Q(A,\vec{x})$ coincides with

$$A \cap \bigcap \big\{\overline{B}_1 \cup \ldots \cup \overline{B}_n \ \big| \ B_1, \ldots, B_n, B'_1, \ldots, B'_m \text{ open rational balls,} \\ B_i \cap B'_j = \emptyset, \ \vec{x} \in B_1, \ A \subseteq B_1 \cup \ldots \cup B'_m\big\} \ . \tag{6}$$

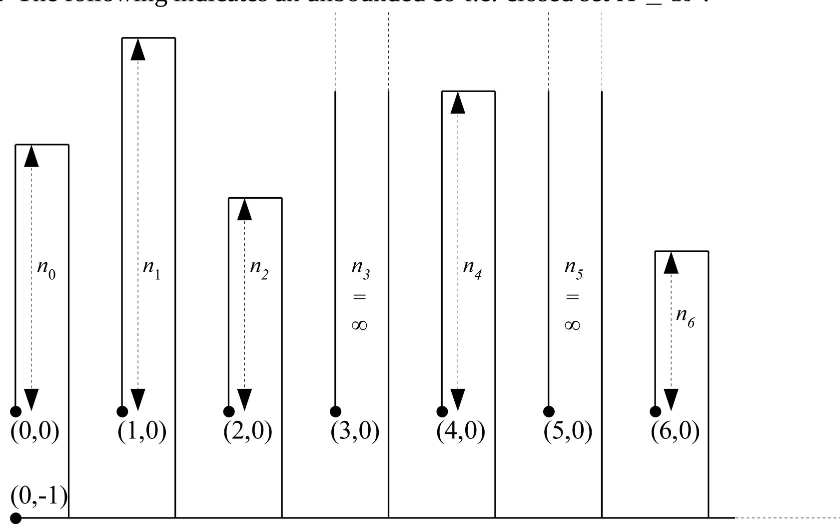
Conditions "$B_i \cap B'_j = \emptyset$" and "$\vec{x} \in B_1$" are semi-decidable; and so is "$A \subseteq B_1 \cup \ldots \cup B'_m$", see for example [Zieg04, LEMMA 4.1b]. Hence $Q(A,\vec{x})$ is $\psi^d_>$–computable via the intersection (6) by virtue of the countable variant of [Weih00, THEOREM 5.1.13.2], compare [Weih00, EXAMPLE 5.1.19.1]. Now finally, $Q(A,\vec{x}) = C(A,\vec{x})$ since components and quasi-components coincide for compact spaces [Kura68, THEOREM §47.II.2].

□

[6] An open subset of Euclidean space is connected if and only if it is path-connected.

Effective boundedness is essential in Proposition 4.1b): one can easily see that $A \mapsto C(A, \vec{x})$ is in general $(\psi^2_>, \psi^2_>)$–discontinuous for fixed computable $\vec{x} \in A$ when a bound on A is unknown. Non-uniformly, we have the following (counter-)

Example 4.2 The following indicates an unbounded co-r.e. closed set $A \subseteq \mathbb{R}^2$:



Here n_e denotes the number of steps performed by the Turing machine with Gödel index e before termination (on empty input), $n_e = \infty$ if it does not terminate (i.e. $e \notin H$).
Consider the connected component C of A with computable handle $(0, -1)$: Were it co-r.e., then one could semi-decide "$(e, 0) \notin C$" [Zieg04, LEMMA 4.1c], equivalently: semi-decide "$e \notin H$": contradiction. □

As opposed to the open case a), a computable 'handle' $\vec{x}$ for a compact connected component $C(A, \vec{x})$ need not exist; hence the non-uniform variant of b) may fail:

Observation 4.3 *A co-r.e. closed subset of $[0, 1]$ obtained from Fact 1.2 has uncountably many connected components, all singletons and none co-r.e.*

Indeed if $A \subseteq [0, 1]$ has positive measure, it must contain uncountably many points x. Each such x is a connected component of its own: otherwise $C(A, x)$ would be a non-empty interval and therefore contain a rational (hence computable) element: contradiction.

Regarding that the counter-example according to Observation 4.3 has uncountably many connected components, it remains to study—in analogy to Section 2.2—the cases of countably infinitely many (Section 4.2) and of

4.1 Finitely Many Connected Components

Does every bounded co-r.e. closed set with *finitely* many connected components have a co-r.e. closed connected component? Proposition 4.1b) stays inapplicable because there still need not exist a computable handle:

Example 4.4 Let $A \subseteq [0, 1]$ denote a non-empty co-r.e. closed set without computable points (recall Fact 1.2). Then $(A \times [0, 1]) \cup ([0, 1] \times A) \subseteq [0, 1]^2$ is (even *path-*) *connected* non-empty co-r.e. closed, devoid of computable points. □

Nevertheless, Proposition 4.5b+c) exhibits a (partial) analog to Corollary 2.9. To this end, observe that a point $\vec{x}$ in some set $A \subseteq \mathbb{R}^d$ is isolated if and only if $\{\vec{x}\}$ is open in A.

Proposition 4.5 *Let $\emptyset \neq A \subseteq \mathbb{R}^d$ be closed.*

a) *If A has finitely many connected components, then each such connected component is open in A.*

b) *If A is co-r.e. and $C(A, \vec{x})$ a bounded connected component of A open in A, then $C(A, \vec{x})$ is also co-r.e.*

c) *If A is r.e. and $C(A, \vec{x})$ a bounded connected component of A open in A, then $C(A, \vec{x})$ is also r.e.*

Corollary 4.6 *If bounded co-r.e. closed $A \subseteq \mathbb{R}^d$ has only finitely many connected components, then each of them is itself co-r.e.*

Proof (Proposition 4.5). a) Let $C(A, \vec{x}_1), \ldots, C(A, \vec{x}_k)$ denote the connected components of A. Each of them is closed. In particular, the finite union $C(A, \vec{x}_2) \cup \ldots \cup C(A, \vec{x}_k)$ is closed and equal to the complement of $C(A, \vec{x}_1)$ in A, hence $C(A, \vec{x}_1)$ is also open in A.

b) As $C(A, \vec{x})$ is open in A, there exists an open subset U of $\mathbb{R}^d$ such that $C(A, \vec{x}) = A \cap U$. Closed $\mathbb{R}^d \setminus U$ being disjoint from closed $C(A, \vec{x})$, there are also disjoint open $V, W \subseteq \mathbb{R}^d$ such that $C(A, \vec{x}) \subseteq V$ and $A \setminus C(A, \vec{x}) \subseteq \mathbb{R}^d \setminus U \subseteq W$ (T_4 separation property, normal space). By compactness, there are finitely many open rational balls $B_1, \ldots, B_n \subseteq V$ covering $C(A, \vec{x})$. Their centers and radii are in particular computable, hence $\overline{B_1} \cup \ldots \cup \overline{B_n}$ is co-r.e. closed. Moreover, as $\overline{B_1} \cup \ldots \cup \overline{B_n} \subseteq \overline{V}$ avoids $W \supseteq A \setminus C(A, \vec{x})$, it holds $C(A, \vec{x}) = A \cap (\overline{B_1} \cup \ldots \cup \overline{B_n})$ which is co-r.e.

c) As in b), $C(A, \vec{x}) = A \cap (B_1 \cup \ldots \cup B_n)$ is closed with r.e. open $B_1 \cup \ldots \cup B_n$, hence $C(A, \vec{x})$ is itself r.e. by by Lemma 4.7 below.

□

Although intersection of closed sets is in general discontinuous [Weih00, THEOREM 5.1.13.3], it holds

Lemma 4.7 *The following mapping is $(\psi^d_<, \theta^d_<, \psi^d_<)$–computable:*

$$\big\{(A, U) : A \subseteq \mathbb{R}^d \textit{ closed}, U \subseteq \mathbb{R}^d \textit{ open}, A \cap U \textit{ closed}\big\} \ \ni\ (A, U) \ \mapsto\ A \cap U \ .$$

Proof. Let A be given as a sequence $(\vec{x}_n)_n \subseteq A$ of real vectors dense in A [Weih00, LEMMA 5.1.10]. Since "$\vec{x}_n \in U$" is semi-decidable [Zieg04, LEMMA 4.1c], one can effectively enumerate (possibly in different order) all those $\vec{x}_n$ belonging to U. Their closure thus lies in $\overline{A \cap U}$ which, by presumption, coincides with $A \cap U$. Conversely, to every $\vec{z} \in A \cap U$, there exists some $\vec{x}_n \in U$ arbitrarily close to $\vec{z}$. We thus conclude that the output subsequence of $(\vec{x}_n)$ is (equivalent to) a valid $\psi^d_<$–name of $A \cap U$. □

4.2 Countably Infinitely Many Connected Components

By Proposition 4.5a+b), if bounded co-r.e. closed $A \subseteq \mathbb{R}^d$ has finitely many components, each one is itself co-r.e. In the case of countably infinitely many connected components, we have seen in Example 4.2 a bounded co-r.e. closed set containing a connected component which is not co-r.e.; others of its components on the other hand are co-r.e. In fact it holds the following counterpart to Fact 2.5b):

Lemma 4.8 *Let $\emptyset \neq A \subseteq \mathbb{R}^d$ be compact with no connected component open in A. Then A has as many connected components as cardinality of the continuum.*

Proposition 4.5b) implies

Corollary 4.9 *Let $A \subseteq \mathbb{R}^d$ be compact and co-r.e. with countable many connected components. Then at least one such component is again co-r.e.*

Proof (Lemma 4.8). By [Kura68, THEOREM §46.V.3], there exists a continuous function $f : A \to \{0, 1\}^\omega$ such that the point inverses $f^{-1}(\bar{\sigma})$ coincide with the quasi-components of A; and these in turn with A's connected components [Kura68, THEOREM §47.II.2]. Since A is compact and f continuous, $f[A] \subseteq \{0, 1\}^\omega$ is compact, too. Moreover every isolated point $\{\bar{\sigma}\}$ of $f[A]$ yields $f^{-1}(\bar{\sigma})$ (closed and) open a component in A. So if A has no open component, $f[A]$ must be perfect—and thus of continuum cardinality by virtue of Fact 2.5b). □

Corollary 4.9 and Example 4.2 leave open the following

Question 4.10 *Is there a* bounded *co-r.e. closed set with countably many connected components, one of which is* not *co-r.e.?*

In view of Proposition 4.1b), this component must not contain a computable point.

4.3 Related Work

An anonymous referee has directed our attention to the following interesting result which appeared as [Mill02, THEOREM 2.6.1]:

Fact 4.11 *For any co-r.e. closed $X \subseteq [0,1]^d$, the following are equivalent:*

(1) *X contains a nonempty co-r.e. closed connected component,*

(2) *X is the set of fixed points of some computable map $g : [0,1]^d \to [0,1]^d$,*

(3) *the image $f(X)$ contains a computable number for any computable $f : X \to \mathbb{R}$.*

5 Co-R.E. Closed Sets *with* Computable Points

The co-r.e. closed subsets of $\mathbb{R}$ devoid of computable points according to Fact 1.2 lack convexity:

Observation 5.1 *Every non-empty co-r.e. interval $I \subseteq \mathbb{R}$ trivially has a computable element:*
Either I contains an open set (and thus lots of rational elements $x \in I$) or it is a singleton $I = \{x\}$, hence x computable [BrWe99, PROPOSITION 3.6].

(It is not possible to continuously 'choose', even in a multi-valued way, some $x \in I$ from a $\psi_>$–name of I, though…) This generalizes to higher dimensions:

Theorem 5.2 *Let $\emptyset \neq A \subseteq \mathbb{R}^d$ be co-r.e. closed and convex. Then there exists a computable point $\vec{x} \in A$.*

Proof. W.l.o.g. suppose A is compact by intersection with some sufficiently large cube $[-N,+N]^d$. Then proceed by induction on d: Under projection $(x_1,\ldots,x_{d-1},x_d) \mapsto x_d$, the image

$$A_d \quad := \quad \big\{x_d \;\big|\; \exists x_1,\ldots,x_{d-1} \in \mathbb{R} : (x_1,\ldots,x_{d-1},x_d) \in A\big\} \quad \subseteq \; \mathbb{R}$$

is convex and $\psi_>$–computable by virtue of [Weih00, THEOREM 6.2.4.4], hence contains a computable point $x_d \in A_d$ (Observation 5.1). The intersection $A \cap (\mathbb{R}^{d-1} \times \{x_d\})$ is therefore non-empty, also convex, and $\psi_>^{d-1}$–computable [Weih00, THEOREM 5.1.13.2]; hence it contains a computable point $(x_1,\ldots,x_{d-1})$ by induction hypothesis. Then $(x_1,\ldots,x_{d-1},x_d)$ is a computable element of A. □

5.1 Star-Shaped Sets

A common weakening of convexity is given in the following

Definition 5.3 *A set $A \subseteq \mathbb{R}^d$ is* star-shaped *if there exists a (so-called* star-*) point $\vec{s} \in A$ such that, for every $\vec{a} \in A$, the line segment*[7] *$[\vec{s},\vec{a}] := \{\lambda\vec{s} + (1-\lambda)\vec{a} : 0 \leq \lambda \leq 1\}$ is contained in A.*
The set of star-points *$S(A)$ is the collection of* all *star-points of A.*

So A is convex if and only if $A = S(A)$; A is star-shaped if and only if $S(A) \neq \emptyset$; and star-shape implies (even simply-)connectedness.

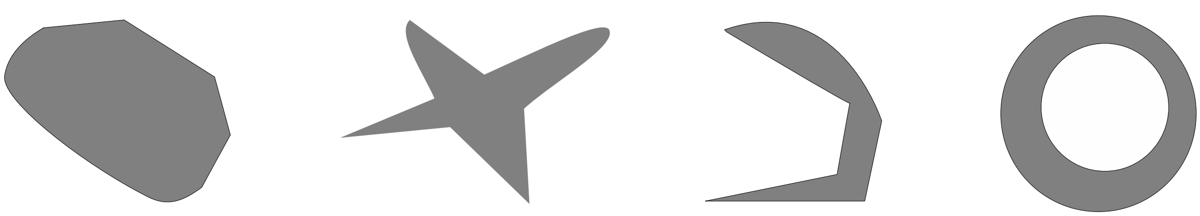


Fig. 1 A convex, a star-shaped, a simply-connected, and a connected set.

Lemma 5.4 *$S(A) \subseteq A$ is convex. Moreover if A is closed, then so is $S(A)$.*

[7] The reader is not in danger of confusing this with the same notion $[\vec{s},\vec{a}]$ standing for the cube $\prod_i [s_i, a_i]$ in Sections 2 and 3.

P r o o f. Let $\vec{x}, \vec{y}$ be star-points of A and $\vec{a} \in A$ arbitrary. By prerequisite the three segments $[\vec{x}, \vec{a}]$ and $[\vec{y}, \vec{a}]$ and $[\vec{x}, \vec{y}]$ all lie in A. Moreover each segment $[\vec{x}, \vec{b}]$ with $\vec{b} \in [\vec{y}, \vec{a}]$—that is the entire closed triangle spanned by $(\vec{x}, \vec{y}, \vec{b})$—also belongs to A; in particular each segment $[\vec{c}, \vec{a}]$ with $\vec{c} \in [\vec{x}, \vec{y}]$ does. Since $\vec{a} \in A$ was arbitrary, this asserts each such $\vec{c}$ to be a star-point of A.

Let $(\vec{x}_n)$ be a sequence of star-points converging to some $\vec{x} \in A$. For arbitrary $\vec{a} \in A$, the segments $[\vec{x}_n, \vec{a}]$ all belong to closed A, hence so does $[\vec{x}, \vec{a}]$. □

Theorem 5.5 *Let $\emptyset \neq A \subseteq \mathbb{R}^2$ be co-r.e. closed and star-shaped. Then A contains a computable point.*

In view of Lemma 5.4 this claim would follow from Theorem 5.2 if, for every star-shaped co-r.e. closed A, its set $S(A)$ of star-points were co-r.e. again. However we have been shown the latter assertion to fail already for very simple compact subsets in 2D [Mill07].

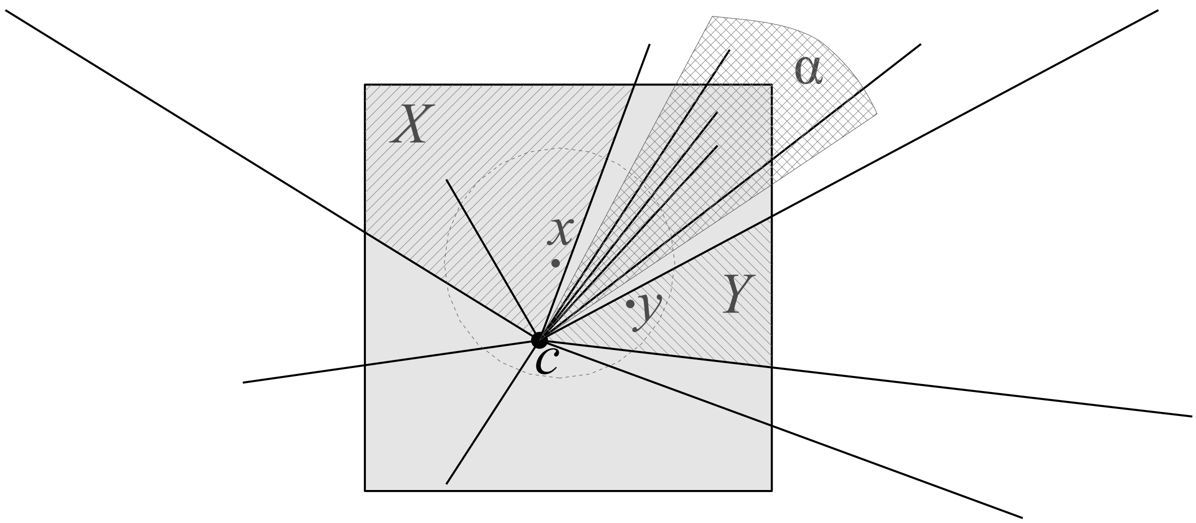


Fig. 2 Illustration to the proof of Theorem 5.5 for the case $S(A) = \{\vec{c}\}$.

Proof (Theorem 5.5). If A has non-empty interior, it contains a rational (and thus computable) point. Otherwise suppose the convex set $S(A)$ to have dimension one, i.e. $S(A) = [\vec{x}, \vec{y}]$ with distinct $\vec{x}, \vec{y} \in A$. Were $S(A)$ a *strict* subset of A, A would contain an entire triangle (compare the proof of Lemma 5.4) contradicting $A^\circ = \emptyset$. Hence $S(A) = A$ is co-r.e. and contains a computable point by Theorem 5.2.

It remains to treat the case of $S(A) = \{\vec{c}\} \subsetneq A$, A consisting of semi-/rays originating from $\vec{c}$ as indicated in Figure 2. Consider some rational square Q containing $\vec{c}$ in its interior but not the entire A. If the square's boundary, intersected with A, contains an isolated point, this point will be computable according to [Weih00, THEOREM 5.1.13.2] and Section 2.2. Otherwise $Q^\circ \setminus A$ consists of uncountably many (Observation 2.4) connected components. Let X and Y denote two *non*-adjacent ones of them, each r.e. open according to Proposition 4.1a). Also let $0 < \alpha \leq 180°$ be some (w.l.o.g. rational and thus computable) lower bound on the angle between X and Y. Notice that X and Y 'almost touch' (i.e. their respective closures meet) exactly in the sought point $\vec{c}$. Moreover for $\vec{x} \in X$ and $\vec{y} \in Y$, elementary trigonometry confirms that $\|\vec{x} - \vec{c}\|_2 \leq \|\vec{x} - \vec{y}\|_2 / (2 \sin \frac{\alpha}{2})$. Based on effective enumerations of all rational $\vec{x} \in X$ and all rational $\vec{y} \in Y$, we thus obtain arbitrary good approximations to $\vec{c}$. □

Regarding a further weakening from convex over star shape, we ask

Question 5.6 *Does every simply-connected co-r.e. closed non-empty subset of $[0,1]^2$ contain a computable point?*

Mere connectedness is not sufficient: recall Example 4.4. This immediately extends to a (counter-)example giving a negative answer to Question 5.6 in 3D:

Example 5.7 Let $A \subseteq [0,1]$ denote a non-empty co-r.e. closed set without computable points. Then $(A \times [0,1]^2) \cup ([0,1] \times A \times [0,1]) \cup ([0,1]^2 \times A) \subseteq [0,1]^3$ is simply-connected non-empty co-r.e. closed devoid of computable points. □

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